INTEGRAL GEOMETRY UNDER G_2 AND $\mathrm{Spin}(7)$

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Abstract. A Hadwiger-type theorem for the exceptional Lie groups G_2 and $\mathrm{Spin}(7)$ is proved. The algebras of G_2 or $\mathrm{Spin}(7)$ invariant, translation invariant continuous valuations are both of dimension 10. Geometrically meaningful bases are constructed and the algebra structures are computed. Finally, the kinematic formulas for these groups are determined.

1. Introduction

Let V be a finite-dimensional vector space. By $\mathcal{K}(V)$ we denote the space of compact convex subsets of V. This space has a natural topology, the Hausdorff topology. A (convex) valuation is a functional $\mu : \mathcal{K}(V) \to \mathbb{C}$ with the following additivity property:

$$\mu(K \cup L) = \mu(K) + \mu(L) - \mu(K \cap L), \quad \forall K, L, K \cup L \in \mathcal{K}(V).$$

Examples of valuations are the Euler characteristic χ (which equals 1 for non-empty K) and any Lebesgue measure.

In this paper, we will only consider continuous and translation invariant valuations. We set $\mathrm{Val} = \mathrm{Val}(V)$ for the vector space of all continuous and translation invariant valuations on V. Alesker [5], [6] introduced a product structure on Val (in fact on some dense subspace consisting of *smooth valuations*).

Let V be endowed with a scalar product. For a compact subgroup G of $\mathrm{SO}(V) \cong \mathrm{SO}(n)$, we let $\mathrm{Val}^G \subset \mathrm{Val}$ denote the subspace of G-invariant valuations. Alesker has shown that Val^G is finite-dimensional if and only if G acts transitively on the unit sphere [2], [7]. In this case, Val^G consists of smooth valuations and Val^G is a finite-dimensional algebra.

Borel [14], [15] and Montgomery-Samelson [24] gave a complete classification of connected compact Lie groups acting transitively and effectively on the sphere (compare also [13], 7.13). There are six infinite series

$$\mathrm{SO}(n), \mathrm{U}(n), \mathrm{SU}(n), \mathrm{Sp}(n), \mathrm{Sp}(n) \cdot \mathrm{U}(1), \mathrm{Sp}(n) \cdot \mathrm{Sp}(1) \tag{1}$$

and three exceptional groups

$$\mathrm{G}_2, \mathrm{Spin}(7), \mathrm{Spin}(9). \tag{2}$$

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For each group G in this list, three fundamental problems of integral geometry are as follows.

(i) **Hadwiger-type theorem for** G: determine the dimension of Val^G and give a geometrically meaningful basis of this space.
(ii) Compute the **algebra structure** of Val^G.
(iii) Compute the whole array of **kinematic formulas** and additive kinematic formulas.

The classical case $G = \mathrm{SO}(n)$ was studied by Blaschke, Chern, Santaló, Hadwiger and others. Hadwiger showed that $\dim \mathrm{Val}^{\mathrm{SO}(n)} = n+1$, with a basis given by the intrinsic volumes $\mu_0 = \chi, \ldots, \mu_n = \mathrm{vol}_n$. The graded algebra $\mathrm{Val}^{\mathrm{SO}(n)}$ is isomorphic to $\mathbb{C}[t]/(t^{n+1})$ (where $\deg t = 1$).

For a compact subgroup G of $\mathrm{SO}(n)$, we set $\bar G := G \ltimes V$ with the product of Haar and Lebesgue measure. The principal kinematic formula for $\mathrm{SO}(n)$ is

$$\int_{\overline{\mathrm{SO}(n)}} \chi(K \cap \bar g L) d\bar g = \sum_{k=0}^n \binom{n}{k}^{-1} \frac{\omega_k \omega_{n-k}}{\omega_n} \mu_k(K) \mu_{n-k}(L), \quad K, L \in \mathcal{K}(V).$$

Here and in the following, ω_n is the volume of the n-dimensional unit ball. Higher and additive kinematic formulas are also classical.

Problem (i) for $G = \mathrm{U}(n)$ was solved by Alesker [3], problem (ii) by Fu [17]. In general, the product structure of Val^G determines all kinematic formulas [11], but it may be rather hard to write down explicit formulas. For $G = \mathrm{U}(n)$, this was recently achieved in [12], thus solving problem (iii).

Alesker [4] computed the dimension of $\mathrm{Val}^{\mathrm{SU}(2)}$. The algebra structure was computed in [8]. The kinematic formulas were known before by work of Tasaki [26].

Problems (i)-(iii) for $G = \mathrm{SU}(n)$ were recently solved in [9] and we will use these results in an essential way in the present work.

For the other groups from list (1)-(2), very few is known.

In this paper, we will solve problems (i)-(iii) for the exceptional Lie groups G_2 and $\mathrm{Spin}(7)$. Our method uses the inclusions

$$\mathrm{SU}(3) < \mathrm{G}_2 < \mathrm{Spin}(7), \mathrm{SU}(4) < \mathrm{Spin}(7)$$

and the fact that the integral geometry of $\mathrm{SU}(3)$ and $\mathrm{SU}(4)$ is well-understood. In particular, we will need that G_2 and $\mathrm{Spin}(7)$ act 2-transitively on the unit sphere. This is not the case for $G = \mathrm{Spin}(9)$, which is the reason why other ideas have to be used in the study of the integral geometry under this group.

2. Results

We define the group G_2 following [16]. Other references are [19], [20] and [18]. A nice historical account can be found in [1].

Let V be a 7-dimensional real vector space. Given $\phi \in \Lambda^3 V^*$, we set

$$b(x,y) := \frac{1}{6} i_x \phi \wedge i_y \phi \wedge \phi \in \Lambda^7 V^*, \quad x, y \in V.$$

If $b(x,x) \neq 0$ for all $x \neq 0$, ϕ will be called *positive*.

Choosing a basis $\omega_1,\ldots,\omega_7$ of V^*, we obtain an example of a positive 3-form by setting

$$\begin{aligned}\phi &= \omega_1 \wedge (\omega_2 \wedge \omega_3 + \omega_4 \wedge \omega_5 + \omega_6 \wedge \omega_7)\\ &\quad + \omega_2 \wedge \omega_4 \wedge \omega_6 - \omega_2 \wedge \omega_5 \wedge \omega_7 - \omega_3 \wedge \omega_4 \wedge \omega_7 - \omega_3 \wedge \omega_5 \wedge \omega_6.\end{aligned}$$

Definition 2.1. *Let $\phi \in \Lambda^3 V^*$ be positive. Then $\mathrm{G}_2 = \mathrm{G}_2(V,\phi)$ is defined as*

$$\mathrm{G}_2 := \{g \in GL(V) | g^*\phi = \phi\}.$$

Since ϕ is positive, $b(x,x)$ lies in the same connected component of $\Lambda^7 V^* \setminus \{0\}$ for all $x \neq 0$. We fix the orientation on V in such a way that $b(x,x) > 0$ for all $x \neq 0$.

Given an orientation preserving isomorphism $\tau : \Lambda^7 V^* \to \mathbb{R}$, we obtain a scalar product on V by setting $\langle x,y\rangle := \tau \circ b(x,y)$. This scalar product induces an isomorphism $\tau' : \Lambda^7 V^* \to \mathbb{R}$, sending ω to $\omega(e_1,\ldots,e_7)$ for a positively oriented orthonormal basis $e_1,\ldots,e_7$ of V. In general $\tau' \neq \tau$. However, there is a unique choice of τ such that $\tau' = \tau$. Indeed, rescaling τ by a positive factor t^2 will rescale τ' by t^{-7}, hence there is a unique t with $\tau' = \tau$. We fix in the following the scalar product with this property.

Since b is G_2-equivariant, G_2 preserves $\langle \cdot,\cdot\rangle$ and the orientation. In other words, $\mathrm{G}_2 \subset \mathrm{SO}(V)$. In particular, the intrinsic volumes $\mu_k, k = 0,\ldots,7$ are G_2-invariant.

Let $W \subset V$ be a 3-dimensional subspace. Choose vectors $w_1, w_2, w_3 \in W$ such that $\|w_1 \wedge w_2 \wedge w_3\| = 1$. We set $\phi(W) := \phi(w_1,w_2,w_3)$. Clearly, $\phi(W)$ is well-defined (i.e. independent of the choice of w_1,w_2,w_3) up to a sign; hence $\phi(W)^2$ is well-defined. It may be checked that two spaces $W_1, W_2 \in \mathrm{Gr}_3(V)$ are in the same G_2-orbit if and only if $\phi(W_1)^2 = \phi(W_2)^2$, but we will not use this fact in the following.

For a face F of a convex polytope $P \subset V$, we let W_F be the linear space parallel to F. The normalized volume of the outer angle at F is denoted by $\gamma(F)$.

Theorem 2.2. *(Hadwiger-type theorem for G_2)*
There is a unique valuation $\nu_3 \in \mathrm{Val}^{\mathrm{G}_2,+}$ such that for each polytope $P \subset V$

$$\nu_3(P) = \sum_{F, \dim F = 3} \gamma(F)\phi(W_F)^2 \operatorname{vol}(F). \tag{3}$$

There is a unique valuation $\nu_4 \in \mathrm{Val}^{\mathrm{G}_2,+}$ such that for each polytope $P \subset V$

$$\nu_4(P) = \sum_{F, \dim F = 4} \gamma(F)\phi(W_F^\perp)^2 \operatorname{vol}(F). \tag{4}$$

The valuations $\mu_0,\ldots,\mu_7,\nu_3,\nu_4$ form a basis of the vector space $\mathrm{Val}^{\mathrm{G}_2}$. In particular,

$$\dim \mathrm{Val}^{\mathrm{G}_2} = 10;$$

and all G_2-invariant valuations are even.

Corollary 2.3. *Let G be any of the groups from the list* (1)-(2)*. Then* Val^G *consists only of even valuations.*

Proof. If $-1 \in G$, then the result is clear. The only groups with $-1 \notin G$ are $SO(n)$ and $SU(n)$ for n odd and G_2. The case $G = SO(n)$ is the classical Hadwiger theorem, which shows in particular that $SO(n)$-invariant valuations are even. The geometric reason behind this is a lemma of Sah ([22], Prop. 8.3.1).

The case $G = SU(n)$ was studied in [9], again there are no odd invariant valuations. Finally, the case $G = G_2$ is treated in Theorem 2.2. □

We set

$$\nu_3' := 5\nu_3 - \mu_3$$
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Theorem 2.4. *(Algebra structure of* $\mathrm{Val}^{\mathrm{G}_2}$*)*
Let t and u be variables of degree 1 *respectively* 3*. Then the map* $t \mapsto \frac{2}{\pi}\mu_1, u \mapsto \frac{\nu_3'}{\pi^2}$ *covers an isomorphism between graded algebras*

$$\mathrm{Val}^{\mathrm{G}_2} \cong \mathbb{C}[t,u]/(t^8, t^2u, u^2 + 4t^6).$$

By the methods of [11], one can translate this algebra structure into the whole array of kinematic formulas for G_2. As an example, we write down in explicit form the principal kinematic formula for G_2.

Corollary 2.5. *(Principal kinematic formula for* G_2*)*
For compact convex sets $K, L \subset V$ we have

$$\int_{\bar{\mathrm{G}_2}} \chi(K \cap \bar{g}L) d\bar{g} = \int_{\overline{\mathrm{SO}(V)}} \chi(K \cap \bar{g}L) d\bar{g} + \frac{1}{2^9}\nu_3'(K)\nu_4'(L) + \frac{1}{2^9}\nu_4'(K)\nu_3'(L).$$

Now we turn our attention to Spin(7). We can define it in a short way as the universal covering of SO(7). However, it will be convenient to have an explicit description similar to the one for G_2 given above.

Let V_+ be a complex 4-dimensional hermitian vector space. Let Ω be the symplectic 2-form of V_+ and $\beta \in \Lambda^4 V_+^*$ be the real part of the holomorphic volume form, i.e. $\beta(v_1, v_2, v_3, v_4) = Re \det(v_1, v_2, v_3, v_4)$. Set

$$\Phi := \frac{1}{2}\Omega \wedge \Omega + \beta \in \Lambda^4 V^*. \tag{5}$$

Definition 2.6.

$$\mathrm{Spin}(V_+) := \{g \in Gl(V_+) | g^*\Phi = \Phi\}.$$

This group is a compact connected subgroup of $\mathrm{SO}(V_+)$ which acts transitively on the unit sphere of V_+. For $v \in S(V_+)$, the stabilizer of $\mathrm{Spin}(V_+)$ at v is $\mathrm{G}_2(W, \phi)$, where $W := T_v S(V_+)$ and

$$\phi := *_W(\Phi|_W) \in \Lambda^3 W^*. \tag{6}$$

For $W \in \mathrm{Gr}_4(V_+)$ we set $\Phi(W) := \Phi(w_1, w_2, w_3, w_4)$, where $w_1, w_2, w_3, w_4 \in W$ satisfy $\|w_1 \wedge w_2 \wedge w_3 \wedge w_4\| = 1$. Then $\Phi(W)$ is well-defined up to a sign. It is easy to show that $\mathrm{Spin}(V_+)$ acts transitively on $\mathrm{Gr}_k(V_+), k \neq 4$ and that $W_1, W_2 \in \mathrm{Gr}_4(V_+)$ belong to the same $\mathrm{Spin}(V_+)$-orbit if and only if $\Phi(W_1)^2 = \Phi(W_2)^2$.

Theorem 2.7. *(Hadwiger-type theorem for* Spin(7)*)*
There exists a unique valuation $\eta \in \mathrm{Val}^{\mathrm{Spin}(V_+)}$ such that for each polytope $P \subset V_+$ we have

$$\eta(P) = \sum_{F, \dim F = 4} \gamma(F) \operatorname{vol}(F) \Phi(W_F)^2. \tag{7}$$

The valuations $\mu_0, \ldots, \mu_8, \eta$ form a basis of $\mathrm{Val}^{\mathrm{Spin}(V_+)}$, in particular

$$\dim \mathrm{Val}^{\mathrm{Spin}(V_+)} = 10.$$

We set $\eta' := 5\eta - \mu_4$. It will turn out that η' is *primitive* in the sense that $\mu_1 \cdot \eta' = 0$.

Theorem 2.8. *(Algebra structure of $\mathrm{Val}^{\mathrm{Spin}(7)}$)*
Let V_+ be, as before, a hermitian vector space of complex dimension 4. Let t and u be variables of degree 1 respectively 4. Then the map $t \mapsto \frac{2}{\pi}\mu_1, u \mapsto \frac{\eta'}{\pi^2}$ covers an isomorphism between graded algebras

$$\mathrm{Val}^{\mathrm{Spin}(V_+)} \cong \mathbb{C}[t, u]/(t^9, u^2 - t^8, ut).$$

Again, we can compute all kinematic formulas for $\mathrm{Spin}(V_+)$ using this theorem and the results of [11]. We only give the following example.

Corollary 2.9. *(Principal kinematic formula for* Spin(7)*)*
For compact convex sets $K, L \subset V_+$ we have

$$\int_{\overline{\mathrm{Spin}(V_+)}} \chi(K \cap \bar{g}L) d\bar{g} = \int_{\overline{\mathrm{SO}(V_+)}} \chi(K \cap \bar{g}L) d\bar{g} + \frac{3}{7!} \eta'(K) \eta'(L).$$

3. Tools

Let V be a vector space of dimension n. Let $\mathrm{Val} := \mathrm{Val}(V)$ be the vector space of continuous, translation invariant valuations. A valuation $\mu \in \mathrm{Val}$ has degree k if $\mu(tK) = t^k \mu(K)$ for all $t \geq 0$. It is even (resp. odd) if $\mu(-K) = \mu(K)$ (resp. $\mu(-K) = -\mu(K)$). We write $\mathrm{Val}_k^{\pm}$ for the subspace of valuations of degree k and parity $\pm$.

Theorem 3.1. *(P. McMullen, [23])*
There is a direct sum decomposition

$$\mathrm{Val} = \bigoplus_{\substack{k=0,\ldots,n \\ \epsilon = \pm}} \mathrm{Val}_k^{\epsilon}.$$

Suppose that V is endowed with a Euclidean scalar product. Let G be a compact subgroup of $\mathrm{SO}(V)$ acting transitively on the unit sphere. Alesker has shown that $\dim \mathrm{Val}^G < \infty$. He introduced on Val^G a structure of a commutative and associative algebra with unit χ satisfying Poincaré duality [5, 6].

The next theorem is a particular case of the more general hard Lefschetz theorem of [10] (where G-invariance is replaced by smoothness in the sense of [6]).

Theorem 3.2. *(Hard Lefschetz theorem,* [10]*)*
Let $\tilde\Lambda : \mathrm{Val}_*^G(V) \to \mathrm{Val}_{*-1}^G(V)$ *be defined by*

$$\tilde\Lambda\mu(K) := \left.\frac{d}{dt}\right|_{t=0} \mu(K+tB).$$

Then $\tilde\Lambda^{2k-n} : \mathrm{Val}_k^G(V) \to \mathrm{Val}_{n-k}^G(V)$ *is an isomorphism. In particular, the Betti numbers*

$$h_k := \dim \mathrm{Val}_k^G$$

satisfy the inequalities

$$h_0 \leq h_1 \leq \ldots \leq h_{\lfloor n/2 \rfloor}$$
$$h_{\lfloor n/2 \rfloor} \geq h_{\lfloor n/2 \rfloor+1} \geq \ldots \geq h_n$$

and the equations

$$h_k = h_{n-k}, \quad k = 0, \ldots, n.$$

The theorem in the case of *even* valuations was proved before by Alesker [3], using the following fundamental result.

Theorem 3.3. *(Klain,* [21]*)*
Let $\mu \in \mathrm{Val}^+(V)$ *be simple, i.e.* μ *vanishes on lower-dimensional compact convex bodies. Then* μ *is a multiple of the Lebesgue measure on* V*, in particular* μ *is of degree* n*.*

This theorem has the following important consequence. The restriction of $\mu \in \mathrm{Val}_k^+$ to a subspace $W \in \mathrm{Gr}_k(V)$ is a multiple of the Lebesgue measure on W. Putting Kl_μ for the proportionality factor, one gets a function $\mathrm{Kl}_\mu \in C(\mathrm{Gr}_k V)$, called the **Klain function** of μ. The resulting map $\mathrm{Val}_k^+ \to C(\mathrm{Gr}_k V)$ is injective.

If μ is smooth in the sense of [6] (in particular if $\mu \in \mathrm{Val}^G$ for one of the groups from (1)-(2)), then there exists a unique valuation $\hat\mu \in \mathrm{Val}_{n-k}^+$ whose Klain function is given by $\mathrm{Kl}_{\hat\mu} = \mathrm{Kl}_\mu \circ \perp$. This valuation is called **Akesker dual** or **Fourier transform** of μ. The Fourier transform can be extended to odd valuations, but we will not need this here.

An analogue of Klain's injectivity result in the case of *odd* valuations was proved by Schneider.

Theorem 3.4. *(Schneider, [25])*
Let $\mu \in \mathrm{Val}^-(V)$ be simple. Then there exists an odd continuous function f on the unit sphere $S(V)$ such that

$$\mu(K) = \int_{S(V)} f(v) dS_{n-1}(K, v),$$

where $S_{n-1}(K, \cdot)$ is the $n-1$-th surface area measure of K. In particular, μ is of degree $n-1$.

Corollary 3.5. *Let V be a Euclidean vector space of dimension n. Let $G < \mathrm{SO}(V)$ act transitively on the unit sphere. For a hyperplane $W \subset V$, let $H := \mathrm{Stab}_G(W)$. Consider the restriction map*

$$\begin{aligned} r_k^\pm : \mathrm{Val}_k^{G,\pm}(V) &\to \mathrm{Val}_k^{H,\pm}(W) \\ \mu &\mapsto \mu|_W. \end{aligned}$$

Then r_k^+ is injective for $k \neq n$ while r_k^- is injective for $k \neq n-1$.

Proof. Since G acts transitively on $\mathrm{Gr}_{n-1}(V)$, a valuation $\mu \in \ker r_k^\pm$ is simple. The statement thus follows from Theorems 3.3 and 3.4. □

If H acts transitively on the unit sphere of W, then the dimension of $\mathrm{Val}_k^{H,\pm}$ is finite and we get a bound for the dimension of $\mathrm{Val}_k^{G,\pm}$. This is the case precisely when G acts 2-transitively on the unit sphere $S(V)$ (this means that if $v_1, v_2, v_1', v_2' \in S(V)$ with $\langle v_1, v_2\rangle = \langle v_1', v_2'\rangle$, then there exists $g \in G$ with $gv_1 = v_1'$ and $gv_2 = v_2'$). For the groups G from the list (1)-(2), only $\mathrm{SO}(n)$, G_2 and Spin(7) have this property.

We will need some results from [12] and [9]. Let V be a hermitian vector space of complex dimension n and $W \in \mathrm{Gr}_k(K)$. Set $p := \lfloor k/2 \rfloor$. The restriction of the symplectic form Ω of V to W can be written in the form

$$\Omega|_W = \sum_{i=1}^p \cos(\theta_i) \alpha_{2i-1} \wedge \alpha_{2i}.$$

Here $\alpha_1, \ldots, \alpha_k$ is an orthonormal basis of the dual space W^* and $0 \leq \theta_1 \leq \ldots \leq \theta_p \leq \frac{\pi}{2}$ is called the **multiple Kähler angle** of W [27]. We denote the q-th elementary symmetric function by σ_q.

Theorem 3.6. *([12])*
For $0 \leq q \leq p$, there exists a unique valuation $\tau_{k,q} \in \mathrm{Val}_k^{\mathrm{U}(V)}$ such that for a polytope $P \subset V$

$$\tau_{k,q}(P) = \sum_{F, \dim F = k} \gamma(F) \operatorname{vol}(F) \sigma_q(\cos^2 \theta_1(W_F), \ldots, \cos^2 \theta_p(W_F)).$$

These valuations are called **Tasaki valuations**. Note that $\tau_{k,0} = \mu_k$, the k-th intrinsic volume.

Under the subgroup $\mathrm{SU}(V) < \mathrm{U}(V)$, there are some more invariant valuations; they are even and appear in the middle degree n. More precisely,

$$\begin{aligned} \mathrm{Val}_k^{\mathrm{SU}(V)} &= \mathrm{Val}_k^{\mathrm{U}(V)} \quad \text{if } k \neq n; \\ \dim \mathrm{Val}_n^{\mathrm{SU}(V)} &= \mathrm{Val}_k^{\mathrm{U}(V)} + 4 \quad \text{if } n \equiv 0 \mod 2; \\ \dim \mathrm{Val}_n^{\mathrm{SU}(V)} &= \mathrm{Val}_k^{\mathrm{U}(V)} + 2 \quad \text{if } n \equiv 1 \mod 2. \end{aligned}$$

Let us describe the "new" invariant valuations more explicitly.

For $W \in \mathrm{Gr}_n(V)$, we set $\Theta(W) := \det(w_1, \ldots, w_n)$, where $w_1, \ldots, w_n$ is a basis of W with $\|w_1 \wedge \ldots \wedge w_n\| = 1$. If $\Omega|_W$ is non-degenerated (which happens if and only if $n = 2p$ is even and $\theta_p(W) < \frac{\pi}{2}$), we ask in addition that $w_1, \ldots, w_n$ be a positively oriented basis.

If $\Omega|_W$ is non-degenerated, then $\Theta(W) \in \mathbb{C}$ is a well-defined invariant. Otherwise, $\Theta(W) \in \mathbb{C}/\{\pm 1\}$ is well-defined. It is easily checked (compare [9]) that in both cases

$$|\Theta(W)| = \prod_{i=1}^p \sin \theta_i(W). \tag{8}$$

Theorem 3.7. *([9])*
There exists a unique valuation $\phi_{n,2} \in \mathrm{Val}_n^{\mathrm{SU}(V)}$ such that

$$\phi_{n,2}(P) = \sum_{F, \dim F = n} \mathrm{vol}(F) \gamma(F) \Theta(W_F)^2.$$

If $n = 2p$ is even, there exists a unique valuation $\phi_{n,1} \in \mathrm{Val}_n^{\mathrm{SU}(V)}$ such that

$$\phi_{n,1}(P) = \sum_{F, \dim F = n} \mathrm{vol}(F) \gamma(F) \Theta(W_F) \prod_{j=1}^p \cos \theta_j(W_F).$$

Note that these valuations are complex-valued. They were denoted by ϕ_1, ϕ_2 in [9].

4. Construction of invariant valuations

In this section, we let V_+ be a complex 4-dimensional hermitian vector space with the 4-form Φ as in (5) and $V \subset V_+$ a hyperplane with the positive 3-form ϕ from (6). We will write G_2 instead of $\mathrm{G}_2(V, \phi)$.

By Corollary 3.5 with $G := \mathrm{G}_2$ and $H := \mathrm{SU}(3)$, we get the following dimensions:

$$\begin{aligned} &\dim \mathrm{Val}^{\mathrm{G}_2,-} = 0 \\ &\dim \mathrm{Val}_k^{\mathrm{G}_2,+} = 1 \quad k \neq 3, 4 \\ &\dim \mathrm{Val}_3^{\mathrm{G}_2,+} = \dim \mathrm{Val}_4^{\mathrm{G}_2,+} \leq 2. \end{aligned}$$

In particular, there are no non-zero odd G_2-invariant valuations, which is non-trivial since $-1 \notin \mathrm{G}_2$.

Clearly, there are no odd $\mathrm{Spin}(V_+)$-invariant valuations, since $-1 \in \mathrm{Spin}(V_+)$. We apply Corollary 3.5 with $G := \mathrm{Spin}(V_+)$ and $H := \mathrm{G}_2$ and find the following dimensions:

$$\begin{aligned}\dim \mathrm{Val}_k^{\mathrm{Spin}(V_+)} &= 1, \quad k \neq 4\\ \dim \mathrm{Val}_4^{\mathrm{Spin}(V_+)} &\leq 2.\end{aligned}$$

Let us show that in fact $\dim \mathrm{Val}_4^{\mathrm{Spin}(V_+)} = 2$, which also implies that $\dim \mathrm{Val}_3^{\mathrm{G}_2} = \dim \mathrm{Val}_4^{\mathrm{G}_2} = 2$.

Recall that $\mathrm{SU}(V_+)$ is a subgroup of $\mathrm{Spin}(V_+)$. Besides the intrinsic volume μ_4, there is another $\mathrm{SU}(V_+)$-invariant valuation of degree 4 which remains invariant under the bigger group $\mathrm{Spin}(V_+)$.

Proposition 4.1. $\eta := \frac{1}{2}\tau_{4,0} - \frac{1}{2}\tau_{4,1} + \frac{3}{2}\tau_{4,2} + \frac{1}{2}Re\phi_{4,2} + 2Re\phi_{4,1} \in \mathrm{Val}_4^{\mathrm{SU}(V_+)}$ *is* $\mathrm{Spin}(V_+)$*-invariant and satisfies* (7).

Proof. Let $W \in \mathrm{Gr}_4(V)$. Fix a basis $w_1, \ldots, w_4$ of W with $\|w_1 \wedge \ldots \wedge w_4\| = 1$. If $\Omega|_W$ is non-degenerated, we ask in addition that $w_1 \wedge \ldots \wedge w_4$ be positive.

Let us write $\theta_i := \theta_i(W), i = 1, 2$; $\sigma_q := \sigma_q(\cos\theta_1, \cos\theta_2), q = 0, 1, 2$ and $\Theta := \Theta(W)$. Then, by (5) and (8),

$$\begin{aligned}\Phi(W)^2 &= (\cos\theta_1 \cos\theta_2 + Re\Theta)^2\\ &= \cos^2\theta_1 \cos^2\theta_2 + (Re\Theta)^2 + 2\cos\theta_1\cos\theta_2 Re\Theta\\ &= Re\left(\cos^2\theta_1\cos^2\theta_2 + \frac{1}{2}\Theta^2 + \frac{1}{2}sin^2\theta_1 \sin^2\theta_2 + 2\cos\theta_1\cos\theta_2\Theta\right)\\ &= \frac{1}{2}\sigma_0 - \frac{1}{2}\sigma_1 + \frac{3}{2}\sigma_2 + Re\left(\frac{1}{2}\Theta^2 + 2\cos\theta_1\cos\theta_2\Theta\right).\end{aligned}$$

Using Theorems 3.6 and 3.7 we get for each polytope $P \subset V_+$

$$\eta(P) = \sum_{F, \dim F = 4} \mathrm{vol}(F)\gamma(F)\Phi(W_F),$$

which is (7). This equation shows in particular that η is $\mathrm{Spin}(V_+)$ invariant. $\square$

Theorem 2.7 follows from the proposition.

Let V be a hyperplane in V_+. The stabilizer of $\mathrm{Spin}(V_+)$ at V is $\mathrm{G}_2 = \mathrm{G}_2(V, \phi)$, where $\phi := *_W \Phi|_W$ (note that ϕ is only well-defined up to a sign depending on the choice of orientation on V).

We set $\nu_4 := \eta|_V \in \mathrm{Val}_4^{\mathrm{G}_2}$. From (6) we deduce that

$$\phi(W^\perp)^2 = \Phi(W)^2, \quad W \in \mathrm{Gr}_4(V).$$

Equation (4) follows immediately.

We define ν_3 by

$$\nu_3 := \hat{\nu}_4 \in \mathrm{Val}_3^{\mathrm{G}_2}.$$

Unfortunately, the verification of (3) seems to be less simple. If a continuous translation invariant valuation satisfies (3), then clearly its Klain

function is given by $W \mapsto \phi(W)^2$, which is also the Klain function of $\nu_3 = \hat\nu_4$. Without giving the details, we indicate that (3) follows by noting that ν_3 is a constant coefficient valuation in the sense of [12]. In fact, if $N_1(K) \in \mathcal{I}_7(V \times V)$ denotes the disk bundle of K, then we can represent ν_3 by

$$\nu_3(K) = \frac{1}{2\pi^2} N_1(K)(p_1^*\phi \wedge p_2^* * \phi).$$

Here $p_1, p_2 : V \times V \to V$ are the natural projections. Note that $p_1^*\phi \wedge p_2^* * \phi \in \Lambda^7(V^* \times V^*)$ has constant coefficients.

5. Algebra structure and kinematic formulas

Proof of Theorem 2.8. Since $\mathrm{SU}(V_+) \subset \mathrm{Spin}(V_+)$, we get a canonical injection of graded algebras

$$\mathrm{Val}^{\mathrm{Spin}(V_+)} \hookrightarrow \mathrm{Val}^{\mathrm{SU}(V_+)}.$$

The image of η in $\mathrm{Val}^{\mathrm{SU}(V_+)}$ is $\frac12\tau_{4,0} - \frac12\tau_{4,1} + \frac32\tau_{4,2} + \frac12 Re\phi_{4,2} + 2Re\phi_{4,1}$. The valuations $\phi_{4,2}$ and $\phi_{4,1}$ are in the annihilator of μ_1. Using the results of [12] and [9], one computes that

$$\begin{aligned}
\mu_1 \cdot \eta &= \mu_1 \cdot \left(\frac12\tau_{4,0} - \frac12\tau_{4,1} + \frac32\tau_{4,2}\right) = \frac{8}{15}\mu_5,\\
\eta \cdot \eta &= \left(\frac12\tau_{4,0} - \frac12\tau_{4,1} + \frac32\tau_{4,2} + \frac14\phi_{4,2} + \frac14\bar\phi_{4,2} + \phi_{4,1} + \bar\phi_{4,1}\right)^2\\
&= \left(\frac12\tau_{4,0} - \frac12\tau_{4,1} + \frac32\tau_{4,2}\right)^2 + \frac18\phi_{4,2}\bar\phi_{4,2} + 2\phi_{4,1}\bar\phi_{4,1}\\
&= \frac{203}{3}\mu_8.
\end{aligned}$$

Setting $\eta' := 5\eta - \mu_4$ one gets $\eta' \cdot \mu_1 = 0$ and $\eta' \cdot \eta' = \frac{7!}{3}\mu_8$. □

Next, we study the algebra structure of $\mathrm{Val}^{\mathrm{G}_2}$. Fix a hyperplane $V_0 \subset V$. Let $x \in V_0^\perp$ be of norm 1. Then $i_x\phi$ is a symplectic form on V_0 and there exists a unique almost complex structure J on V_0 with $\langle v, w\rangle = i_x\phi(Jv, w)$.

The stabilizer of G_2 at V_0 is given by $\mathrm{SU}(V_0) \cong \mathrm{SU}(3)$.

Lemma 5.1. *Let $r : \mathrm{Val}^{\mathrm{G}_2} \to \mathrm{Val}^{\mathrm{SU}(V_0)}$ be the natural restriction map. Then*

$$\begin{aligned}
r(\nu_3) &= \frac12\tau_{3,0} - \frac12\tau_{3,1} + \frac12 Re\phi_{3,2}\\
r(\nu_4) &= \tau_{4,2}.
\end{aligned}$$

Proof. For $W \in \mathrm{Gr}_3(V_0)$, set $\theta := \theta_1(W)$ and $\Theta := \Theta(W)$. Then $\phi(W) = Re\Theta(W)$ and therefore

$$\phi(W)^2 = (Re\Theta)^2 = \frac12 Re(\Theta^2 + |\Theta|^2) = \frac12 Re(\Theta^2 + 1 - \cos^2\theta).$$

For $W \in \mathrm{Gr}_4(V_0)$, we have $\phi(W^\perp) = \cos\theta_2 = \sigma_2(\cos\theta_1, \cos\theta_2)$, where $0 = \theta_1 \leq \theta_2$ are the Kähler angles of W.

The statement of the lemma now follows from the definitions in Theorem 3.6 and Theorem 3.7. □

Proof of Theorem 2.4. The Alesker product is compatible with the restriction $r : \mathrm{Val}^{\mathrm{G}_2} \to \mathrm{Val}^{\mathrm{SU}(V_0)}, \mu \mapsto \mu|_{V_0}$, hence

$$r(\mu_2 \cdot \nu_3) = \frac{1}{2}\mu_2 \cdot (\tau_{3,0} - \tau_{3,1}) = \frac{4}{5}\mu_5.$$

Similarly,

$$\begin{aligned} r(\nu_3 \cdot \nu_3) &= \left(\frac{1}{2}\tau_{3,0} - \frac{1}{2}\tau_{3,1} + \frac{1}{4}\phi_{3,2} + \frac{1}{4}\bar{\phi}_{3,2}\right)^2 \\ &= \frac{1}{4}(\tau_{3,0} - \tau_{3,1})^2 + \frac{1}{8}\phi_{3,2}\bar{\phi}_{3,2} \\ &= -\frac{9\pi}{8}\mu_6. \end{aligned}$$

Since r_5^+ and r_6^+ are injective, we deduce that $\mu_2 \cdot \nu_3 = \frac{4}{5}\mu_5$ and $\nu_3 \cdot \nu_3 = -\frac{9\pi}{8}\mu_6$ in $\mathrm{Val}^{\mathrm{G}_2}$. From these two equations, the theorem follows. □

Proof of Corollary 2.5. Let us write ν_4 in terms of the basis $\mu_4, \mu_1 \cdot \nu_3$ of $\mathrm{Val}_4^{\mathrm{G}_2}$. As above, we get that $r(\mu_1 \cdot \nu_3) = \frac{3}{16}\pi(\mu_4 - \tau_{4,2})$ and therefore

$$\nu_4 = \mu_4 - \frac{16}{3\pi}\mu_1 \cdot \nu_3.$$

Corollary 2.5 now follows from the methods in [11]. □

We rescale the derivation operator $\tilde{\Lambda}$ of Theorem 3.2 as in [12] by setting, on $\mathrm{Val}_k^{G_2}$,

$$\Lambda := \frac{\omega_{7-k}}{\omega_{8-k}}\tilde{\Lambda}$$

and rescale multiplication by μ_1 by

$$L\mu := \frac{2\omega_k}{\omega_{k+1}}\mu_1 \cdot \mu.$$

Corollary 5.2. *The operators L, Λ define, together with the degree counting operator $H\mu = (2\deg\mu - 7)\mu$ a representation of the Lie algebra $\mathfrak{sl}_2$ on $\mathrm{Val}^{\mathrm{G}_2}$.*

Proof. The commutator relations $[H, L] = 2L$ and $[H, \Lambda] = -2\Lambda$ are immediate. We have to show that $[L, \Lambda]\mu = H\mu$ for each $\mu \in \mathrm{Val}^{\mathrm{G}_2}$. If μ is one of the intrinsic volumes, then this follows from a direct computation. It remains to show that this equation also holds for $\mu = \nu_3$ and $\mu = \nu_4$. The Fourier transform intertwines L and Λ (compare [11]) and therefore

$$\Lambda\nu_3 = \Lambda\hat{\nu}_4 = \widehat{L\nu_4} = \hat{\mu}_5 = \mu_2.$$

Similarly,

$$\Lambda\nu_4 = \Lambda\hat{\nu}_3 = \widehat{L\nu_3} = \hat{\mu}_4 - \hat{\nu}_4 = \mu_3 - \nu_3.$$

It is now easy to compute that $[L,\Lambda]\nu_3 = -\nu_3$ and $[L,\Lambda]\nu_4 = \nu_4$. $\square$

Let us conclude with some general remarks. The algebra structure of the spaces Val^G is now known for the groups $G = \mathrm{SO}(n), \mathrm{U}(n), \mathrm{SU}(n), \mathrm{G}_2, \mathrm{Spin}(7)$ from the list (1)-(2) . By looking at these algebras, some natural questions arise.

- For G as above, rescaled versions of the derivation operator and of multiplication by μ_1 define an $\mathfrak{sl}_2$-representation on Val^G. We conjecture that this will also be the case for the remaining groups from the list (1)-(2). On the bigger space Val^{sm} of smooth translation invariant valuations, this is not true.
- By Corollary 2.3, $\mathrm{Val}^G \subset \mathrm{Val}^+$ for each of these groups G. However, our proof relies on a non-geometric case by case argument. Is there a direct explanation of this fact?
- For each such group G, Val^G consists of constant coefficient valuations in the sense of [12]. Is this also true for the remaining groups from the list (1)-(2)?

References

[1] Agricola, I.: Zur Geschichte der Ausnahme-Lie-gruppe G_2. *Mitteilungen der Deutschen Mathematikervereinigung* **15** (2007), 242–247.
[2] Alesker, S.: On P. McMullen's conjecture on translation invariant valuations. *Adv. Math.* **155** (2000), 239–263.
[3] Alesker, S.: Hard Lefschetz theorem for valuations, complex integral geometry, and unitarily invariant valuations. *J. Differential Geom.* **63** (2003), 63–95.
[4] Alesker, S.: SU(2)-invariant valuations. *Geometric aspects of functional analysis. Papers from the Israel seminar (GAFA) 2002-2003.* LNM 1850 (2004), 21–29.
[5] Alesker, S.: The multiplicative structure on polynomial valuations. *Geom. Funct. Anal.* **14** (2004), 1–26.
[6] Alesker, S.: Theory of valuations on manifolds I. Linear spaces. *Israel J. Math.* **156** (2006), 311–339.
[7] Alesker, S.: Valuations on manifolds: a survey. To appear in *Geom. Funct. Anal.*
[8] Bernig, A.: A product formula for valuations on manifolds with applications to the integral geometry of the quaternionic line. Preprint arXiv:math/0611264.
[9] Bernig, A.: A Hadwiger-type theorem for the special unitary group. Preprint arXiv:0801.1606.
[10] Bernig, A., Bröcker, L.: Valuations on manifolds and Rumin cohomology. *J. Differential Geom.* **75** (2007), 433–457.
[11] Bernig, A., Fu. J. H. G.: Convolution of valuations. *Geom. Dedicata* **123** (2006), 153–169.
[12] Bernig, A., Fu, J. H. G.: Hermitian integral geometry. Preprint arXiv:0801.0711.
[13] Besse, A. L.: *Einstein manifolds.* Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, Bd. 10. Berlin etc.: Springer-Verlag (1987).
[14] Borel, A.: Some remarks about Lie groups transitive on spheres and tori. *Bull. Am. Math. Soc.* **55** (1949), 580–587.
[15] Borel, A.: Le plan projectif des octaves et les sphères comme espaces homogènes. *C. R. Acad. Sci.* **230** (1950), 1378–1380.
[16] Bryant, R. L.: Metrics with exceptional holonomy. *Ann. Math.* **126** (1987), 525–576.
[17] Fu, J. H. G.: Structure of the unitary valuation algebra. *J. Differential Geom.* **72** (2006), 509–533.

[18] Fulton, W., Harris, J.: *Representation theory. A first course.* Graduate Texts in Mathematics 129. Springer, New York 1991.
[19] Hitchin, N.: The geometry of three-forms in six dimensions. *J. Differ. Geom.* **55** (2000), 547–576.
[20] Joyce, D. D.: *Compact manifolds with special holonomy.* Oxford Mathematical Monographs. Oxford: Oxford University Press 2000.
[21] Klain, D.: A short proof of Hadwiger's characterization theorem. *Mathematika* **42** (1995), 329–339.
[22] Klain, D., Rota, G.-C.: *Introduction to Geometric Probability.* Lezione Lincee, Cambridge University Press 1997.
[23] McMullen, P.: Valuations and Euler-type relations on certain classes of convex polytopes. *Proc. London Math. Soc.* **35** (1977), 113–135.
[24] Montgomery, D., Samelson, H.: Transformation groups of spheres. *Ann. Math.* **44** (1943), 454–470.
[25] Schneider, R.: Simple valuations on convex bodies. *Mathematika* **43** (1996), 32–39.
[26] Tasaki, H.: Integral geometry under the action of the first symplectic group. *Arch. Math.* **80** (2003), 106–112.
[27] Tasaki, H.: Generalization of Kähler angle and integral geometry in complex projective spaces II. *Math. Nachr.* **252** (2003), 106–112.

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